Exercice 1. The transfer map. Let $\alpha\colon H \hookrightarrow G$ be a subgroup and M a $\mathbb{Z}G$-module (which can be viewed as a $\mathbb{Z}H$-module). We continue the particular case of Sheet 3, Exercise 5, Point 5.

1. When H has finite index in G, show that $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} M$ as $\mathbb{Z}G$-modules.
2. When H has finite index in G, compute the composite
$$M \to \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} M \to M$$
3. When H has finite index in G, compute $H_*(G; \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M))$ and deduce from this the existence of homomorphisms $\mathrm{tr}_H^G\colon H_*(G;M) \to H_*(H;M)$. Dualize briefly.
4. Show that the composition $\alpha_* \circ \mathrm{tr}_H^G$ is multiplication by the index. We only ask for the homological case here.
5. When G is a finite group, prove that $|G| \cdot H_k(G;M) = 0$ for $k > 0$ and any $\mathbb{Z}G$-module M.
6. Under the same assumptions as in Point 5, conclude that if $|G|$ is invertible in M, then $H_k(G;M) = 0$ for $k > 0$.

$\Diamond$ **Exercice 2. The norm map.** Let C_n be a cyclic group of order n, and t a generator. Let M be a left $\mathbb{Z}C_n$-module and $N\colon M \to M$ be the multiplication by the sum $1 + t + \cdots + t^{n-1}$ of all elements in C_n.

1. Identify the cochain complex $\mathrm{Hom}_{\mathbb{Z}C_n}(F_\bullet, M)$ where $F_\bullet$ is the periodic resolution from Week 1 (the map N appears there !).
2. Show that N induces a map $\overline{N}\colon M_{C_n} \to M^{C_n}$, called the *norm map.*
3. Show that $H^{2n}(C_n;M) \cong M^G/(N \cdot M) \cong \mathrm{Coker}\,\overline{N}$ for $n \geq 1$.
4. Show that all odd cohomology groups are isomorphic to the kernel of $\overline{N}$.
5. Identify all cohomology groups of C_n with trivial coefficients $\mathbb{Z}$ and $\mathbb{F}_p$.
6. Prove that $\mathrm{Coker}\,\overline{N}$ and $\ker \overline{N}$ are annihilated by n (use a map $M^{C_n} \to M_{C_n}$).

$\Diamond$ **Exercice 3. Shapiro Lemma.** Let $H < G$ be a subgroup, M be a left $\mathbb{Z}H$-module and define the induced module $\mathrm{Ind}_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M$ and the coinduced module $\mathrm{Coind}_H^G M = \mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, M)$, where $\mathbb{Z}G$ is seen as right or left module over $\mathbb{Z}H$ via group multiplication.

1. Prove that $\mathrm{Ind}_H^G M$ contains M as a submodule over $\mathbb{Z}H$ and decomposes as a direct sum of the gM's where g ranges over a set of representatives for G/H.
2. Let M be the permutation module $\mathbb{Z}[G/H]$, where G acts by left multiplication on the cosets. Show that it is isomorphic to $\mathrm{Ind}_H^G \mathbb{Z}$.
3. Prove that $H_n(H;M) \cong H_n(G; \mathrm{Ind}_H^G M)$.
4. Prove that $H^n(H;M) \cong H^n(G; \mathrm{Coind}_H^G M)$.
5. Show that induced modules from the trivial subgroup $1 < G$ are acyclic in homology.

◊ **Exercice 4. Semi-direct products.** Let G be a group, fix a $\mathbb{Z}G$-module M, and consider an extension $0 \to M \xrightarrow{i} E \xrightarrow{\pi} G \to 1$.

1. Show that the extension splits if and only if there is a subgroup $H < E$ such that $E = i(M) \cdot H$ and $i(M) \cap H = \{1\}$.
2. Show that the extension splits if and only if there is a subgroup $H < E$ such that every element of E can be expressed uniquely as $e = i(m)h$ for $m \in M, h \in H$.
3. Show that the extension splits if and only if it is equivalent to the semi-direct product extension $M \rtimes G$.
4. Show that the symmetric group S_3 is a semi-direct product $\mathbb{Z}/3 \rtimes C_2$. Compute $H^1(C_2; \mathbb{Z}/3)$ for the corresponding $\mathbb{Z}C_2$-module structure on $\mathbb{Z}/3$ and find all conjugacy classes of sections of the extension $\mathbb{Z}/3 \to S_3 \to C_2$.
5. Same exercice as point 4 for the product $\mathbb{Z}/2 \times C_2$.
6. **Bonus** : Same exercise as point 4 for the unitriangular matrix group in $GL_3(\mathbb{F}_3)$ as a semi-direct product $(\mathbb{Z}/3 \times \mathbb{Z}/3) \rtimes C_3$. This is the subgroup of upper triangular matrices with 1's on the diagonal. It has order 27 and all elements except 1 have order 3 (prove this).

◊ **Exercice 5. Hochschild cohomology.** Let k be a commutative ring, A a (unital) k-algebra with multiplication μ, and M an A-bimodule (the left and right actions commute : $a(mb) = (am)b$). We write $\otimes$ for the tensor product of k-modules. Recall Hochschild homology from exercise 5, Week 1.

1. Define A^{op} as the *opposite* k-algebra where the product $a \cdot b = ba$. We write $A^e = A \otimes A^{\mathrm{op}}$. Show that a left A^{op}-module is the same as a right A-module, and thus an A^e-module is an A-bimodule.
2. Define the bar complex $C_\bullet^{bar}(A)$ with $C_n(A) = A^{\otimes n+2}$ (with augmentation given by μ). Show that $HH_n(A; M)$ is the homology of the chain complex $M \otimes_{A^e} C_\bullet^{bar}(A)$.
3. Define $HH^n(A, M)$ as the cohomology of the cochain complex $\mathrm{Hom}(C_\bullet^{bar}(A), M)$. Identify this with the cohomology of $\mathrm{Hom}_k(A^\bullet, M)$ (write down the differentials of this cochain complex).
4. Identify $HH^0(A; M)$ with the invariants M^A (define them).
5. Identify $HH^1(A; M)$ with the quotient of the k-module of derivations $Der(A, M)$ by the submodule of inner derivations (and define them).

◊ indicates the weekly assignments.